CHAPTER 3

No Non-Trivial Poisson Process is at the Same Time a Non-Trivial Bernoulli Process

At the time of writing, there is standardisation action to rewrite IEC 61508-7:2016 Annex D, “A probabilistic approach to determining software safety integrity for pre-developed software”, because of misleading information which it contains. During the course of this action, some participants have proposed connecting the mathematics of Bernoulli Processes with those of Poisson Processes, because of mathematical similarities: in some respects the mathematics of Poisson Processes is that obtained in the limit if the number of Bernoulli trials N tends to infinity. I point out here what experienced statistical practitioners already know, that you cannot connect the two types of process deterministically. In contrast to the arguments used by statisticians, the arguments here are purely structural, and therein lies any novelty they may have.

Consider a process of sequential trials as in a Bernoulli Process, but without the condition that the probability of failure per trial is constant and not dependent on the results of previous trials. Call this a Pseudobernoulli process. (It is a characterisation of an on-demand process, but the term “on demand” carries intellectual baggage which I would rather avoid here.)

3.1 Is Digital Control Fundamentally “Pseudobernoulli”?

There is also an argument which I have heard from various colleagues which says that any digital system has fundamentally discrete operation, and any continuous process, such as that modelled by a Poisson Process, is implemented as a series of very rapid discrete demands. Thus such a process represents also a rapidly repetitive Pseudobernoulli Process represented by processor cycles or some abstraction of them. There arises a suggestion that the two cases need not be distinguished; alternatively that a Poisson Process is "really" a Pseudobernoulli Process in the digital world. We shall see below that construing a Poisson Process as Pseudobernoulli does not bring much if any intellectual leverage, and indeed seems likely to lead to loss of information. Then we shall see that the Pseudobernoulli Process cannot be a Bernoulli Process.

It is true that nominally-continuous real-world processes are often well approximated by rapidly-cycling digital discrete processes, otherwise digital computers would not be so useful in system control. To react "continuously" to a parameter value, two digital processes (or actions) are generally involved, which can be distinguished:

sampling a process samples the value (reads the sensor input) at high frequency (hundreds to thousands to millions of Hertz, as necessary) and passes on those values which need a control reaction to the control process;

control a control process inputs the values forwarded from the sample process which need control reaction, calculates that control action and issues an appropriate command to actuators which effect that control.

These two processes might well be implemented in the same piece of software code, or they may be distinguished, depending on the control architecture. The control action is an "on-demand" process: on a demand (generated by the forwarding action of the sampling process) it calculates and issues a control reaction. As seen from "outside" the implementation, the dynamic control process overall may well fit the Poisson continuous Process model. This nominally-Poisson Process is implemented as two actions, one of which, the control action, is on-demand. That on-demand action would be appropriately modelled as a Pseudobernoulli Process. So far so good. One may investigate the mathematical connection between the "outside view" Poisson Process and the Pseudobernoulli process, the control action. One can attempt to relate the Pseudobernoulli demands on the control action to the time at which those demands occur. This is called a process in statistics, and I shall call it SP1 for historical reasons. The question is

- whether SP1 is in some sense deterministic, in which case explicit mathematics can possibly relate the Pseudobernoulli Process constituted by the control action to the Poisson Process approximated by the sample and control actions;

- or whether SP1 is generally stochastic.

In the first case, we would have a functional relationship between the Pseudobernoulli Process of the control action and the Poisson Process represented by the "outside view". Intuitively, SP1 is a process that reacts to parameters which derive from features in the world changing values over a threshold sufficient for the sample action to pass those values to the control action. We might well expect that the general nature of such a process is stochastic rather than functional.

3.2 A Poisson Process as Pseudobernoulli

Suppose a given process P has outcomes 0 or 1, or, as they are usually called for these processes, $\mathit{Failure}$ and $\mathit{Success}$. Suppose also that we wish to consider this process under two different models, as a Poisson Process and as a Pseudobernoulli Process.

P as a Pseudobernoulli Process has

1. a parameter n = the number of trials, and
2. an outcome of $\mathit{Success}$ or $\mathit{Failure}$ per trial.

As described, a Pseudobernoulli process is not a stochastic process. There are no stochastic parameters associated with it. It has been defined purely structurally. Were P to be a Bernoulli Process, it would also fulfil the condition that

- there is a fixed probability p that a given trial will end in $\mathit{Failure}$.

This is of course a stochastic condition. We shall see later that this condition cannot coherently be superimposed on a Pseudobernoulli Process arising from a Poisson Process.

I denote the result of trial t in a Pseudobernoulli Process by $\mathit{OutcomeB}(t)$. $\mathit{OutcomeB}$ is thus a function from the set of trials to $\{\mathit{Success}, \mathit{Failure}\}$. For a Bernoulli Process, it is important to note that the probability of outcome of the next trial, p of $\mathit{Failure}$ and $(1 - p)$ of $\mathit{Success}$, is independent of any result of any previous trial: a Bernoulli Process is said to be memoryless.

P as a Poisson Process has

1. a parameter t = elapsed time,
2. a probability $q(t_i)$ that there will be a Failure outcome in the next time interval

T_i of length t_i (that is, the probability of failure in T_i is solely dependent upon its length, a property which is also called memoryless-ness),

3. an outcome function $OutcomeP(t)$ in which $OutcomeP(t) = Success$ when $OutcomeP(t) \neq Failure$, and in which the set

$$FAIL = \{t \,:\, OutcomeP(t) = Failure\}$$

is a collection of left-closed right-open intervals. Each interval represents an initial point of failure, the left point, and a time to repair, namely the length of the interval; the absent right point represents the time at which success recommenced. We shall abide by the convention that $OutcomeP(0) = Success$.

Extended models consider also the lengths of these intervals, yielding various parameters concerned with time to repair. These will not concern us. The mathematics of memoryless-ness is also more precise, but the details also need not concern us.

It follows from the fact that the set $FAIL$ is a collection of left-closed right-open intervals, and the observation that $OutcomeP(t) = Success$ when $OutcomeP(t) \neq Failure$, that the set

$$SUCCEED = \{t \,:\, OutcomeP(t) = Success\}$$

is also a collection of left-closed, right-open intervals. I shall speak of a time point t being "in an interval in" $SUCCEED$ or $FAIL$,and mean by that just $t \in SUCCEED$, respectively $t \in FAIL$.

Considering P as a Pseudobernoulli Process means that there is no time parameter. But there is a time parameter when P is considered as a Poisson Process. We can reconcile these two models by considering n (= number of Pseudobernoulli trials) to be a function $k(t)$ of time. We can take $k(t)$ to be the number of Pseudobernoulli trials which have taken place up to and including time t. Observe that $k(t)$ is a monotonically increasing integer function of t. Indeed, a monotonically incrementing integer function of t. We abide without loss of generality by the convention that $k(0) = 0$.

The Pseudobernoulli outcome function $OutcomeB$ is also a discrete function. To turn it into a continuous function of time, we may define $Outcome_B(t)$ to be the real-time function $OutcomeB(k(t))$.

3.3 Co-Interpretation

There is a straightforward way in which the Pseudobernoulli parameters and Poisson parameters for P can be structurally reconciled. Each interval in $SUCCEED$ and in $FAIL$ is left-closed. At each point of

$$SuccessPoints = \{t : t \text{ is the left end−point of an interval in } SUCCEED\}$$

we could consider a Pseudobernoulli trial to have been performed with outcome $Success$. Similarly, at each point of

$$FailPoints = \{t : t \text{ is the left end−point of an interval in } FAIL\}$$

we could consider a Pseudobernoulli trial to have been performed with outcome $Failure$. It is easy to see that a specific discrete process is defined by this interpretation, namely one in which Outcomes strictly alternate:

$$Outcome_B(t) = Failure \text{ when } k(t) \text{ is odd}$$

$$Outcome_B(t) = Success \text{ when } k(t) \text{ is even}$$

This is not a stochastic process. There is no non-trivial probability of failure associated with the next trial. The failure is functional on the previous trial: the probability of the next trial being a failure is 0 in the case that the previous trial resulted in $Failure$, and it is 1 in the case that the previous trial resulted in $Success$. It is also apparent from this that this process is not memoryless (in the stochastic sense): the result of the next trial is functionally dependent upon the result of the last trial. This Pseudobernoulli process is not stochastic; in particular it is not a Bernoulli Process. And it is a minimal Pseudobernoulli Process. Here is an example of what I mean. Consider the Pseudobernoulli Process taking place over the time interval $[0, T]$

$Trial(1)$ takes place at time t_1 and results in $Success$

$Trial(2)$ takes place at time t_2 and results in $Success$

$Trial(3)$ takes place at time t_3 and results in $Failure$

$Trial(4)$ takes place at time t_4 and results in $Success$

$Trial(5)$ takes place at time t_5 and results in $Failure$

$Trial(6)$ takes place at time t_6 and results in $Success$

$Trial(7)$ takes place at time t_7 and results in $Success$

$Trial(8)$ takes place at time t_8 and results in $Failure$

Then

$$SUCCEED = [0,t_3) \cup [t_4,t_5) \cup [t_6,t_8)$$

$$FAIL = [t_3,t_4) \cup [t_5,t_6) \cup [t_8,T]$$

and of course

$$SUCCEED \cup FAIL = [0,T]$$

$$Successpoints = \{t_1, t_4, t_6\}$$

$$Failpoints = \{t_3, t_5, t_8\}$$

Pseudobernoulli trials 2 and 7, at times t_2 and t_7 respectively, have disappeared. The Pseudobernoulli Process resulting from the structural reconstruction given above is just

$Trial(1)$ takes place at time t_1 and results in $Success$

$Trial(3)$ takes place at time t_3 and results in $Failure$

$Trial(4)$ takes place at time t_4 and results in $Success$

$Trial(5)$ takes place at time t_5 and results in $Failure$

$Trial(6)$ takes place at time t_6 and results in $Success$

$Trial(8)$ takes place at time t_8 and results in $Failure$

So yes, we can identify a Pseudobernoulli Process given a Poisson Process (and when we have reason to believe that a Pseudobernoulli Process underlies the Poisson Process), but this reconstruction may miss some Pseudobernoulli trials. If we were to write the trial results as a sequence of 0's (for $Failure$) and 1's (for $Success$), then the original sequence is 11010110 and the reconstructed sequence 101010. The reconstruction as above will always result in an alternating sequence. So we cannot identify a "real" Pseudobernoulli Process underlying a given Poisson Process by being given the Poisson Process; just part of it.

3.4 Characterisation of Joint Processes

Now let us look at the stochastic conditions. The probability of failure is given for the next time interval T in a Poisson process P by $q(length(T))$. We can identify a Pseudobernoulli Process congruent with P as a Poisson process if we have reason to believe that one underlies P, but this Pseudobernoulli Process will be minimal - there might well be trials missing (and generally will be). But suppose we are given a Pseudobernoulli Process underlying P. Can this Pseudobernoulli Process actually be a Bernoulli Process? We shall see that the answer is no.

Let us assume that P is a Poisson Process and that P is also modelling by a Pseudobernoulli Process that is in fact Bernoulli; that is, there is a probability of failure for each trial, this is p, and p is constant, and the trials are memoryless. Let us assume $p \neq 0$ and $p \neq 1$. We shall derive a contradiction.

The quantities $q(length(T))$ and p are both probabilities of failure. One is per-time-interval-length, one is per trial. They both represent the same quantity, the probability of failure, but they are differently parametrised. Without loss of generality, take a length of time $[0, t_{final}]$ long enough that $q(t_{final}) > p$ and let T_0 be a time interval inside $[0, T_{final}]$ such that $q(length(T_0)) = p$. Then $length(T_0)$ is the length of a time interval in which the probability of failure is that of exactly one Bernoulli trial. Indeed, this will be so for any interval T for which $length(T) = length(T_0)$, by memoryless-ness. Let us denote $length(T_0)$ by L_0.

Suppose no Bernoulli trials would be deemed to take place within T_0. Then the probability of failure in T_0 would be 0, but we have assumed it is p and $p \neq 0$. That would be a contradiction. It follows that it cannot be the case that no Bernoulli trials are deemed to take place within T_0.

Suppose we were to deem two Bernoulli trials to take place in T_0. The probability of double failure in T_0, $q(T_0)$, would then be p^2 by the memoryless property of Bernoulli processes. But we have assumed $q(T_0) = p$, and $p \neq 1$ and thus $p^2 \neq p$. $q(T_0)$ cannot be equal to both p^2 and p; that would be a contradiction. It follows that it cannot be the case that two Bernoulli trials are deemed to take place within T_0. Similar reasoning shows that three or more Bernoulli trials cannot be deemed to take place in T_0.

Thus for T of length L and $q(T) = p$, where p is not 0 or 1, it follows that exactly one Bernoulli trial must be deemed to take place in the next L_0 period of time. This constrains the joint interpretation, as we shall see, impossibly strongly.

The $OutcomeP$ function of P as a Poisson Process and the $Outcome_B$ function are functions giving the outcome ($Success$ or $Failure$) of P as a Poisson process and P as a Bernoulli process respectively. Each function must either remain constant through the next time period T of length L_0), or change value precisely once. Hence P as a Poisson process is a process for which, within any time interval T of length L_0, there is either one failure with probability $p = q(T)$ or no failure (with probability $(1 - p)$).

Let us introduce the notation $ProbFail(x)$ to denote the probability of failure of P at some time in x, where x is a time interval. Consider any interval T of length L_0. By the memoryless property, for failure behaviour it suffices to consider the interval $[0,L_0]$. We have shown that there can be at most one failure in this interval. Suppose there is a failure , and it occurs at time $S < L_0$. It follows that $ProbFail((S, L_0])$ is 0, and, by subtracting S from both endpoints, it follows that $ProbFail((0, (L_0 - S)])$ is also 0 by the memoryless property. Since there is not a failure at $t = 0$ (by hypothesis), it follows that

$$ProbFail([0, (L_0 - S)]) = ProbFail([0,0]) + ProbFail((0, (L_0 - S)]) = 0 + 0 = 0$$

It further follows that

$$ProbFail(((L_0 - S), L_0]) = 0 + ProbFail(((L_0 - S), L_0])$$

$$= ProbFail([0, (L_0 - S)]) + ProbFail(((L_0 - S), L_0])$$

$$= ProbFail([0,L_0]) = p$$

Taking $(L_0 - S)$ away from both endpoints of $((L_0 - S), L_0]$ yields the interval $(0, S]$. So, again by the memoryless property,

$$ProbFail((0, S]) = ProbFail(((L_0 - S), L_0]) = p$$

There is never a failure at $t = 0$ by hypothesis, so $ProbFail([0,0]) = 0$, and

$$ProbFail([0, S]) = ProbFail(0) + ProbFail((0, S]) = 0 + p = p$$

Since the probability of failure in the interval $[0, (L_0 - S)]$ is 0, and the probability of failure in the interval $[0, S]$ is p, and we are assuming $p \neq 0$, it follows that

$S > (L_0 - S)$, that is, that $S > \frac{1}{2}L_0$. In other words, any failure must occur in the second half of any interval T of length L_0. It follows that

$$ProbFail([0, \frac{1}{2}L_0]) = 0$$

and

$$ProbFail((\frac{1}{2}L_0, L_0]) = p$$

Again, by memoryless-ness, taking $\frac{1}{2}L_0$ away from both endpoints,

$$ProbFail([\frac{1}{2}L_0, L_0]) = ProbFail([0, \frac{1}{2}L_0]) = 0$$

However,

$$ProbFail([\frac{1}{2}L_0, L_0]) = ProbFail(\frac{1}{2}L_0) + ProbFail((\frac{1}{2}L_0, L_0]) = 0 + p = p$$

But it cannot be both 0 and p, because $p \neq 0$. This is a contradiction.

We have shown by reductio ad absurdum that a Poisson process P cannot also be taken to exhibit Bernoulli trials with probability of failure p where p has any value except 0 or 1. A value of p equal to 0 or 1 yields a trivial Bernoulli Process.

It follows that, although a Poisson Process can be taken to exhibit a Pseudobernoulli Process, such a process can not be a (non-trivial) Bernoulli Process.

3.5 Conclusion

Consider process P to be interpreted as a Poisson Process. If there is reason to think that a Pseudobernoulli Process underlies P, then part of this process may be reconstructed structurally from P, but not necessarily all of it - the reconstructed part will show alternating successes and failures, but consecutive successes and failures will have disappeared. Furthermore, if there is a Pseudobernoulli Process underlying the Poisson Process, this cannot be at the same time a Bernoulli Process except in unenlightening cases.